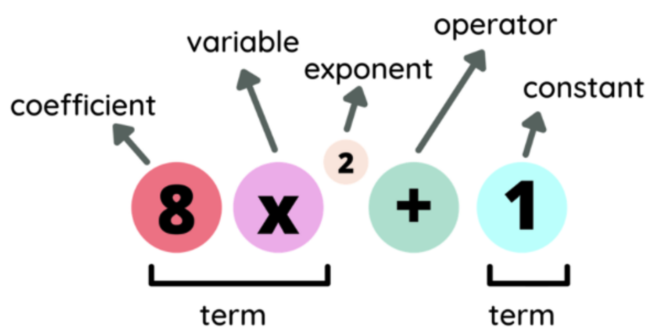
Grade 7/8 Math Circles
March 18-21, 2024
Polynomials

What is a Polynomial?

A polynomial is a mathematical expression that consists only of variables of whole-number exponents, whose coefficients are any real number. A polynomial expression only has addition, subtraction or multiplication between each term with a variable; a polynomial will never have division by its variable.

An example of a polynomial is



The **coefficient** of a term in a polynomial is the constant number multiplying the variable. Here, the coefficient 8 multiplies with the **variable** x, which has an **exponent** of 2. Each term in a polynomial is separated by a plus or minus sign. In this example, the '+' sign separates the first term from the second term, giving us two total terms. Lastly, we say that the **degree** of a polynomial is the value of the variable's largest exponent in all of the polynomial's terms. Since the only term with a variable in the example above is $8x^2$, the degree of the polynomial is the exponent of x^2, which is 2.

Example 1

State the degree and number of terms in the polynomial $x^8 - 3x^5 + x^4 + 6$.

Solution:

There are a total of 4 terms separated by a plus or minus sign. The degree is the largest value of exponent on our variable, x, which is 8. So the polynomial has **4 terms** with **degree 8**.

Combining Terms

To combine the terms of a polynomial, we add and subtract *like-terms*. Like-terms are terms that have a common variable degree. For example, the terms $16x^3$ and $-9x^3$ are like-terms since they both have a degree of 3. To add them, we simply add their coefficients to write $16x^3 - 9x^3 = 7x^3$. Similarly, the terms 55 and 12 are like-terms since they are both constants (degree 0). Adding them is nothing special since $55 + 12 = 67$, but they are nonetheless like-terms. We can now say that the entire expression $16x^3 - 9x^3 + 55 + 12$ is equivalent to $7x^3 + 67$.

Exercise 1

Simplify the polynomial below by combining like-terms.

$$-2x^6 + 10x^2 + 1 - 13x^2 - 3$$

Note: If we're dealing with polynomials of many terms, it is often helpful to order the terms from greatest degree to lowest. This way we can more easily identify like-terms and simplify the expression quicker.

Stop and Think

How can we use polynomials to solve real-world problems? Will we ever be able to graph or solve equations with polynomials?

Polynomial Functions

Recall that a function can be thought of as a machine that takes an input x and spits out an output y. The value of y depends on what the function is. If the function is $y = 3x - 5$, plugging in $x = 2$ will spit out a function value of $y = 3x - 5 = 3(2) - 5 = 6 - 5 = 1$. A polynomial function is just a function whose output value comes from a polynomial. Since $3x - 5$ is a polynomial, the function $y = 3x - 5$ above is considered a polynomial function. It is shown below in Figure 1. Another example is the degree-two polynomial function $y = x^2 - x + 2$ (Figure 2). Finally, a third degree polynomial function could be $y = -x^3 + 4x^2 + 2x - 1$ (Figure 3).

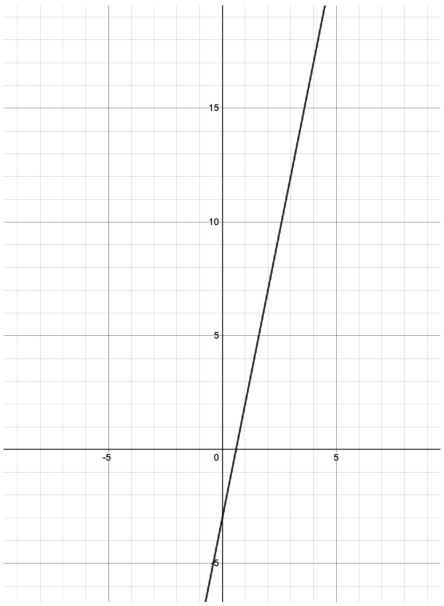


Figure 1: $y = 3x - 5$

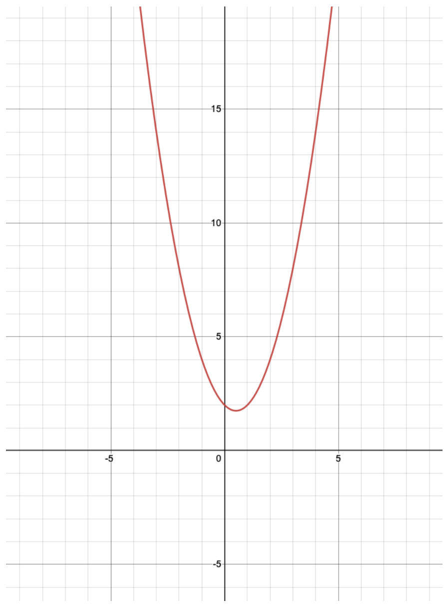


Figure 2: $y = x^2 - x + 2$

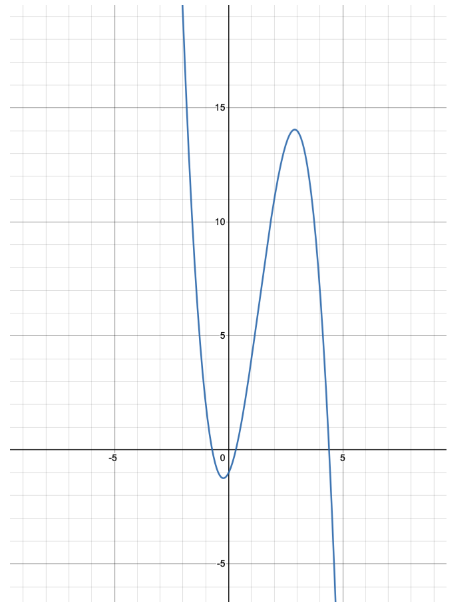


Figure 3: $y = -x^3 + 4x^2 + 2x - 1$

Analysing Degree

We see that the 1^{st}-degree polynomial's graph has 0 turns, the 2^{nd}-degree polynomial's graph has 1 turn, and the 3^{rd}-degree polynomial's graph has 2 turns. You may notice a pattern here. It seems that every time we increase the degree of a polynomial, we increase the number of possible turns that its graph has. This observation is true! If our polynomial has degree n, the maximum number of turns it can have is $n - 1$. It may have less than $n - 1$ turns, however it can *never* have more. Once we learn something called 'calculus', we can actually prove this.

Exercise 2

Experiment graphing different polynomial functions using Desmos. Count the number of turns for each graph, and try to convince yourself that this number will always be less than the degree n. Specifically, the number of turns will always be between 0 and $n - 1$. Can you find a polynomial function of degree n with *less* than $n - 1$ turns?

There are many ways we can categorize polynomial functions without doing any calculations. First, we can look closer at its degree. An *even-degree polynomial* is a polynomial whose degree is an even number. For example, the function $y = x^4 - x + 17$ has degree 4, therefore since 4 is even, its degree is even. Similarly, if a polynomial function has an odd degree like 1, 3 or 5, it is considered an *odd-degree polynomial*.

Example 2

Determine if the function below is odd or even-degree, then comment on the number of turns its graph has.

$$y = 5x^7 + x^4 - 8x^3 + x + 19$$

Solution:

The term with the largest degree is $5x^7$, which has an exponent of 7. Therefore, the polynomial has a degree of 7. 7 is an odd number, which means the polynomial is odd-degree. Further, we predict that the graph of this polynomial function has $n - 1 = 7 - 1 = 6$ turns, although it could potentially have less. We can find out by graphing.

Dominant Term

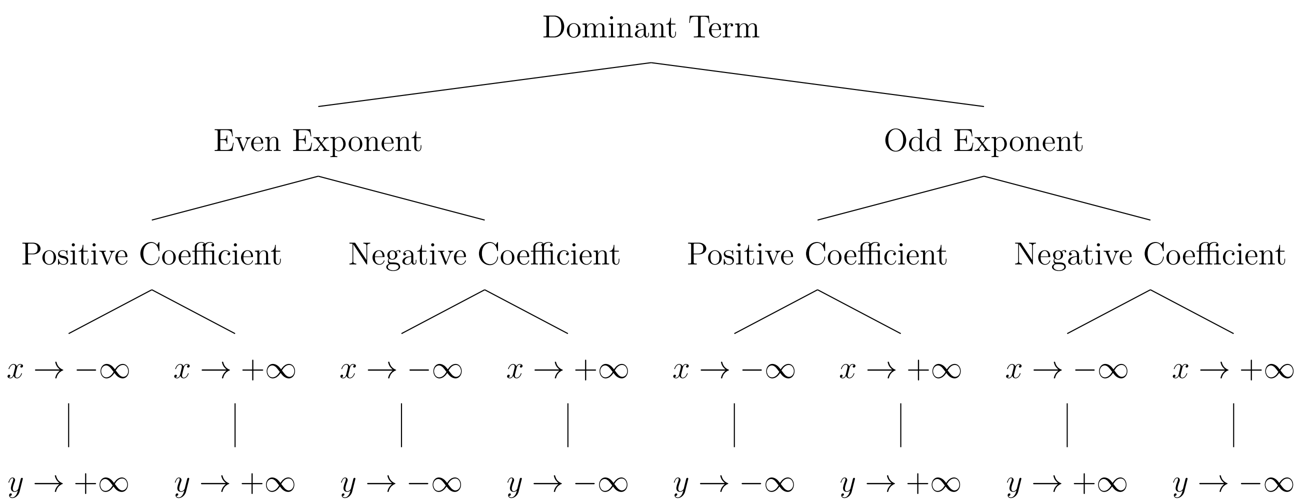
The dominant term of a polynomial is the term with the highest degree. The degree of the dominant term determines the degree and 'end behaviour' of the entire polynomial.

End Behaviour

The end-behaviour of a function is the value it approaches as we make x either largely negative ($x \to -\infty$) or largely positive ($x \to +\infty$). We see in Figure 2 above (an even-degree function) that the function's value gets larger as we move to the left or the right. This is because as the value of x becomes largely positive (like $100,000$) or largely negative (like $-100,000$), the value of the 'dominant' term, x^2, gets extremely large (towards infinity). Although the entire function is $y = x^2 - x + 2$, the value of $-x + 2$ is substantially smaller than the dominant x^2 term when x is sufficiently positive or negative. This means it can essentially be ignored when considering large-scale behaviour of the function.

The end behaviour of a polynomial function only depends on its dominant term!

When analysing end behaviour, we do not care about the function's shape when x is between $-\infty$ and $+\infty$. We only need to look at the dominant term; in particular, we care about its coefficient, and whether its degree is odd or even. The chart below will aid us in determining a polynomial function's end behaviour based on such information.



Example 3

Consider the polynomial function

$$y = 2x^3 - x^2 + 6x - 1$$

(a) What is the dominant term of the polynomial? Is it even or odd-degree?

(b) What is its end behaviour?

Solution:

(a) The dominant term is $2x^3$, so the degree of the polynomial is 3. The degree, 3, is an odd number, therefore the polynomial is odd-degree.

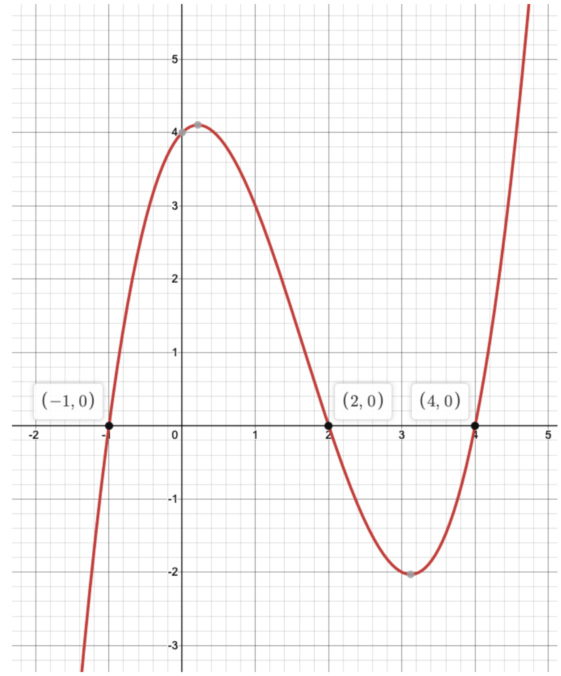
(b) The dominant term $2x^3$ has a positive coefficient $(+2)$, and the polynomial is odd-degree. Referring back to our chart above, we see that an odd-degree polynomial with a positive coefficient on its dominant term has end behaviour $y \to -\infty$ as $x \to -\infty$, and $y \to \infty$ as $x \to \infty$. This is because as x gets very negative, $2x^3$ also becomes very negative. Similarly, when x gets large, $2x^3$ gets large.

Exercise 3

How are the graphs of the functions $y = -7x^4 + 2x - 5$ and $y = -x^6 - 5x^3 - 8x + 6$ similar? How are they different? Explain by comparing the degree, shape, and end behaviour.

Roots of a Polynomial Function

The root of a function is the value of x that produces a function value of 0: its x-intercept. For example, in the graph of $y = \frac{1}{2}x^3 - \frac{5}{2}x^2 + x + 4$ to the right, the function has a height of 0 when $x = -1$, $x = 2$ and $x = 4$. Therefore, the roots of the function are $x = -1,\ 2,\ 4$. If we plugged in any of these values of x into the function, we would get an output of 0. There are many interesting ways to find the roots of polynomial functions, but this may be impossible to do without a computer. For now, we will stick to simpler functions that can be solved using algebra.



The main way we solve for the roots of a function are to set the output y equal to zero and solve for x using algebra. For 1^{st} degree (linear) and 2^{nd} degree (quadratic) polynomials, this can be done by hand. For higher-degree functions, we may need to write a program to find the roots for us.

Example 4

Find and verify the root(s) of the function $y = 18x + 3$, then explain what x represents.

Solution:

To find the roots of any function, we set its output y equal to 0 then solve for x using algebra.

$$y = 18x + 3 \implies 0 = 18x + 3$$

$$0 - 3 = 18x + \cancel{3} - \cancel{3}$$

$$\frac{-3}{18} = \frac{\cancel{18}x}{\cancel{18}} \implies x = \frac{-3}{18} = \frac{-1}{6}$$

This tells us that the only time the function crosses the x-axis is when $x = -\frac{1}{6}$. To verify, we plug $x = -\frac{1}{6}$ into the equation and make sure it gives us 0. It does!

$$y = 18x + 3 = 18 \cdot -\frac{1}{6} + 3 = -\frac{18}{6} + 3 = -3 + 3 = 0$$

Exercise 4

Solve for the roots of the function $y = x^2 - 25$.

The Quadratic Formula

It may be obvious that it's not always easy to solve for roots of a polynomial function. For example, what if we wanted the roots of the function $y = x^2 + 3x - 40$? After we set $y = 0$, we would be stuck. Luckily, there is a tool we can use to let us find the roots of any 2^{nd}-degree polynomial we want: the quadratic formula. The formula is shown below for a general quadratic function $y = ax^2 + bx + c$, where a, b and c represent the constant coefficients of each term.

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that we **always** must make one side of the equation equal to 0! For our example above where $y = x^2 + 3x - 40$, we set $a = 1$, $b = 3$ and $c = -40$. Plugging this into the quadratic formula gives us

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(-40)}}{2(1)} = \frac{-3 \pm \sqrt{169}}{2} = \frac{-3 \pm 13}{2}$$

Solving for each case, we get the two roots are $x = \frac{-3+13}{2} = \frac{10}{2} = 5$ and $x = \frac{-3-13}{2} = \frac{-16}{2} = -8$.

Example 5

Find the roots of $y = 2x^2 - 5x + 3$ using the quadratic formula.

Solution:

We set $a = 2$, $b = -5$ and $c = 3$, then plug these values into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(3)}}{2(2)} = \frac{5 \pm \sqrt{1}}{4} = \frac{5 \pm 1}{4}$$

So the first root of the function is $x = \frac{5+1}{4} = \frac{6}{4} = \frac{3}{2}$, while the second is $\frac{5-1}{4} = \frac{4}{4} = 1$.

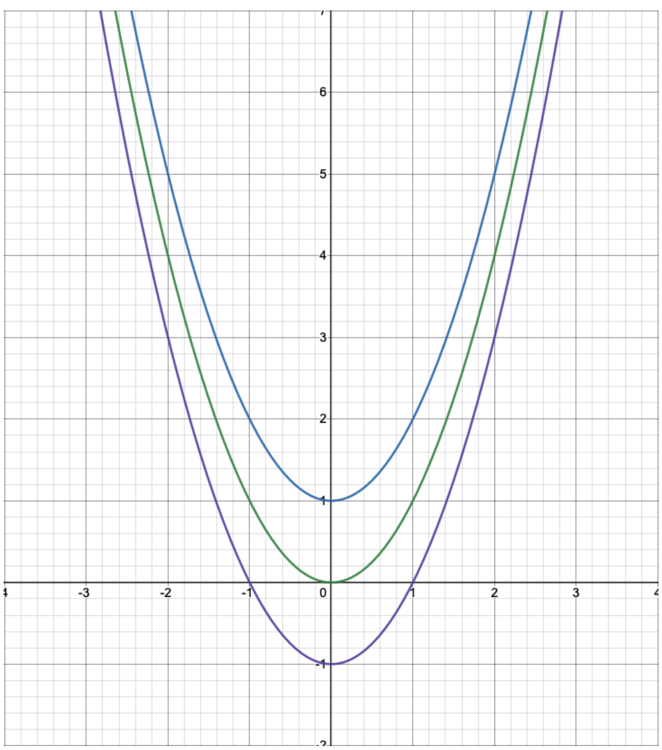
It is natural to wonder if this formula ever fails. Although its result is always true, it does not always give us a *real* answer. In particular, if the number inside the square root ($b^2 - 4ac$) is negative, there will be **no real solutions** since the square root of any negative number is imaginary. We call this number the *discriminant*; it determines whether we have real or imaginary solutions. If the discriminant is positive, there are two real roots of the function. If the discriminant is zero, there is exactly one root. If the discriminant is negative, there are only imaginary roots of the function.

Exercise 5

Determine whether the function $y = x^2 + 1$ has real or imaginary roots.

Visualizing Types of Roots

If a polynomial function y does not have any real roots, that means that there are no real solutions to the equation $y = 0$. However, it can be easier to visualize this on a graph.



Recall that we can imagine the roots of a function as the values of x that give the function y a height of 0. In Exercise 3, we saw the function $y = x^2 + 1$ had no real roots. The graph of $y = x^2 + 1$ on the left (top curve) shows that the curve never reaches the x-axis. Therefore, the function $y = x^2 + 1$ **never** has a value of 0. If we did not have the quadratic formula, we could see by plotting the graph of the function that there are no x-intercepts. Conversely, the bottom curve $y = x^2 - 1$ has two roots ($x = \pm 1$) while the middle function $y = x^2$ has exactly one root at $x = 0$.

Exercise 6

What is the discriminant of the polynomial function $y = 2x^2 - 4x + 2$? What does this tell us?

Example 6

A baseball team is practicing for an upcoming game. When one of the players goes up to bat, they hit the ball far into the outfield at $t = 0$. The ball has a height according to the function $h = -4.9t^2 + 22.5t + 0.4$ for any time t. Find the time at which it hits the ground in the outfield.

Solution:

When the ball hits the ground, its height is zero, so $0 = -4.9t^2 + 22.5t + 0.4$. We set $a = -4.9$, $b = 22.5$ and $c = 0.4$. Now all we need to do is use the quadratic formula to find the time t.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-22.5 \pm \sqrt{(22.5)^2 - 4(-4.9)(0.4)}}{2(-4.9)} = \frac{-22.5 \pm \sqrt{514.1}}{-9.8} = \frac{-22.5 \pm 22.7}{-9.8}$$

The first root of this equation is $t_1 = \frac{-22.5+22.7}{-9.8} \simeq -0.02$ seconds, however the time it hits the ground must be greater than 0, so we use the other solution: $t_2 = \frac{-22.5-22.7}{-9.8} \simeq 4.61$ seconds.

Intersecting Polynomials

Now that we have learned about the characteristics of individual polynomials, we are prepared to solve where multiple polynomials intersect. If we have two functions y_1 and y_2, we can find where they intersect by simply setting the two functions equal to each other.

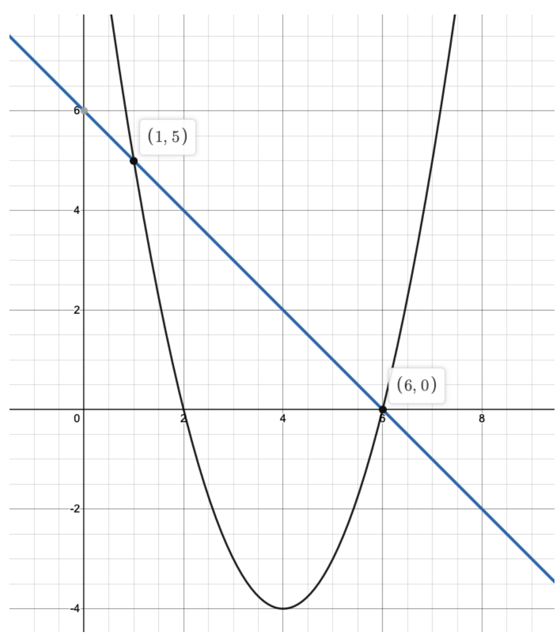
Take the two polynomial functions on the right for example. We have the linear function $y = -x + 6$ intersecting the quadratic function $y = x^2 - 8x + 12$. They intersect at two coordinates: $(1, 5)$ and $(6, 0)$. To solve, we set the right side of each equation equal to the other and solve for x. Here, we set

$$y_1 = y_2 \implies -x + 6 = x^2 - 8x + 12$$

Next, we make one side of the equation equal zero.

$$-\cancel{x} + \cancel{6} + \cancel{x} - \cancel{6} = x^2 - 8x + 12 + x - 6$$

$$0 = x^2 - 7x + 6$$



Now that one side of the quadratic equation is equal to zero, we're ready to use the quadratic formula.

$$0 = x^2 - 7x + 6 \implies a = 1,\ b = -7,\ c = 6$$

Plugging these numbers into the quadratic formula we can find x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\cancel{-}(\cancel{-}7) \pm \sqrt{(-7)^2 - 4(1)(6)}}{2(1)} = \frac{7 \pm \sqrt{25}}{2} = \frac{7 \pm 5}{2}$$

So our two solutions are $x = \frac{7+5}{2} = \frac{12}{2} = 6$ and $x = \frac{7-5}{2} = \frac{2}{2} = 1$. This result matches our graph! If we wanted to verify that $x^2 - 7x + 6 = 0$ when $x = 6$ and $x = 1$, we simply plug in these values of x.

$$\text{For } x = 6:\ \ x^2 - 7x + 6 = (6)^2 - 7(6) + 6 = 36 - 42 + 6 = 0$$

$$\text{For } x = 1:\ \ x^2 - 7x + 6 = (1)^2 - 7(1) + 6 = 1 - 7 + 6 = 0$$

Since these values of x produce a function value of 0, then $x = 6$ and $x = 1$ are both roots of the polynomial function $y = x^2 - 7x + 6$.

Note

When solving the quadratic formula, we set y equal to zero since we are looking for the values of x that produce a value of $y = 0$ from the given function. However, when we solve the equation

$$-x + 6 = x^2 - 8x + 12,$$

we **must** make one side equal to zero in order to use the quadratic formula. Leaving the equation as it is will make it difficult to solve, but moving everything to one side so the other side equals zero lets us use the quadratic formula.

Exercise 7

(a) Give an example of two linear (1^{st}-degree) functions that do not intersect. Why do they not intersect?

(b) Do the same for two quadratic (2^{nd}-degree) functions. Explain your reasoning.

Example 7

Determine when the polynomial function $y = 2x^2 - 3x - 5$ has a height of -3.

Solution: The function $y = 2x^2 - 3x - 5$ has a height of -3 when $y = -3$. Setting the two right-hand sides equal to each other, we get $2x^2 - 3x - 5 = -3$. Adding 3 to both sides, we get

$$2x^2 - 3x - 5 + 3 = \cancel{-3} + \cancel{3}$$

$$2x^2 - 3x - 2 = 0$$

We can now solve for x using the quadratic formula, where $a = 2$, $b = -3$ and $c = -2$. We find the roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-2)}}{2(2)} = \frac{3 \pm \sqrt{9 + 16}}{4} = \frac{3 \pm \sqrt{25}}{4} = \frac{3 \pm 5}{4}$$

We see there are two possible values of x when $y = -3$. Either $x = \frac{3+5}{4} = \frac{8}{4} = 2$, or $x = \frac{3-5}{4} = \frac{-2}{4} = -\frac{1}{2}$.

Formulas for Higher Orders

Just like how there is a quadratic formula for 2nd-degree polynomials, there is a cubic formula for 3rd-degree polynomials. There is also a quartic formula (for 4th-degree), but it is much longer. The cubic formula is shown below for one of three possible roots:

$$x = \sqrt[3]{\left(\frac{bc}{6a^2} - \frac{d}{2a} - \frac{b^3}{27a^3}\right) + \sqrt{\left(\frac{bc}{6a^2} - \frac{d}{2a} - \frac{b^3}{27a^3}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(\frac{bc}{6a^2} - \frac{d}{2a} - \frac{b^3}{27a^3}\right) - \sqrt{\left(\frac{bc}{6a^2} - \frac{d}{2a} - \frac{b^3}{27a^3}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

Clearly, these equations quickly become tedious to solve. After the quadratic formula, it is much easier to solve for the roots of a polynomial using a computer rather than a formula.